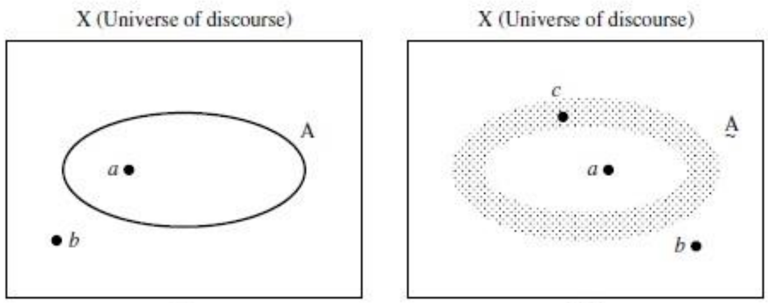
CLASSICAL SETS AND FUZZY SETS

4.1 Introduction

We will describe sets as mathematical abstractions of these events and of the universe itself. Figure 4.1a shows an abstraction of a universe of discourse, say X, and a crisp (classical) set A somewhere in this universe. A classical set is defined by crisp boundaries, i.e., there is no uncertainty in the prescription or location of the boundaries of the set, as shown in Fig. 4.1a where the boundary of crisp set A is an unambiguous line. A fuzzy set, on the other hand, is prescribed by vague or ambiguous properties; hence its boundaries are ambiguously specified, as shown by the fuzzy boundary for set $\underline{A}$ in Fig. 4.1b.

Figure 4.1 again helps to explain this idea, but from a two-dimensional perspective.

Point a in Fig. 4.1a is clearly a member of crisp set A; point b is unambiguously not a member of set A. Figure 4.1b shows the vague, ambiguous boundary of a fuzzy set $\underline{A}$ on the same universe X: the shaded boundary represents the boundary region of $\underline{A}$. In the central (unshaded) region of the fuzzy set, point a is clearly a full member of the set.



(a) (b)

Outside the boundary region of the fuzzy set, point *b* is clearly not a member of the fuzzy set. However, the membership of point *c*, which is on the boundary region, is ambiguous.

If complete membership in a set (such as point *a* in Fig. 4.1*b*) is represented by the number 1, and no-membership in a set (such as point *b* in Fig. 4.1*b*) is represented by 0, then point *c* in Fig. 4.1*b* must have some intermediate value of membership (partial membership in fuzzy set $\underline{A}$ on the interval [0,1]. Presumably the membership of point *c* in $\underline{A}$approaches a value of 1 as it moves closer to the central (unshaded) region in Fig. 4.1*b* of $\underline{A}$and the membership of point *c* in A∼ approaches a value of 0 as it moves closer to leaving the boundary region of $\underline{A}$.

CLASSICAL SETS

Define a universe of discourse, X, as a collection of objects all having the same characteristics.

The individual elements in the universe X will be denoted as *x*. The features of the elements in X can be discrete, countable integers or continuous valued quantities on the real line. Examples of elements of various universes might be as follows:

- The clock speeds of computer CPUs
- The operating currents of an electronic motor
- The operating temperature of a heat pump (in degrees Celsius)
- The Richter magnitudes of an earthquake
- The integers 1 to 10

Most real-world engineering processes contain elements that are real and non-negative. The first four items just named are examples of such elements. However, for purposes of modeling, most engineering problems are simplified to consider only integer values of the elements in a universe of discourse. So, for example, computer clock speeds might be measured in integer values of

megahertz and heat pump temperatures might be measured in integer values of degrees Celsius. Further, most engineering processes are simplified to consider only finite-sized universes. Although Richter magnitudes may not have a theoretical limit, we have not historically measured earthquake magnitudes much above 9; this value might be the upper bound in a structural engineering design problem. As another example, suppose you are interested in the stress under one leg of the chair in which you are sitting. You might argue that it is possible to get an infinite stress on one leg of the chair by sitting in the chair in such a manner that only one leg is supporting you and by letting the area of the tip of that leg approach zero. Although this is theoretically possible, in reality the chair leg will either buckle elastically as the tip area becomes very small or yield plastically and fail because materials that have infinite strength have not yet been developed. Hence, choosing a universe that is discrete and finite or one that is continuous and infinite is a modeling choice; the choice does not alter the characterization of sets defined on the universe. If elements of a universe are continuous, then sets defined on the universe will be composed of continuous elements. For example, if the universe of discourse is defined as all Richter magnitudes up to a value of 9, then we can define a set of "destructive magnitudes," which might be composed (1) of all magnitudes greater than or equal to a value of 6 in the crisp case or (4) of all magnitudes "approximately 6 and higher" in the fuzzy case.
A useful attribute of sets and the universes on which they are defined is a metric known as the cardinality, or the cardinal number. The total number of elements in a universe X is called its cardinal number, denoted n_x, where x again is a label for individual elements in the universe. Discrete universes that are composed of a countably finite collection of elements will have a finite cardinal number; continuous universes comprised of an infinite collection of elements will have an infinite cardinality. Collections of elements within a universe are called sets, and collections of elements within sets are called subsets. Sets and subsets are terms that are often used synonymously, since any set is also a subset of the universal set X. The collection of all possible sets in the universe is called the *whole set*.
For crisp sets A and B consisting of collections of some elements in X, the following notation is defined:

$$x \in X \ \rightarrow x \ belongs \ to \ X$$
$$x \in A \ \rightarrow x \ belongs \ to \ A$$
$$x \notin A \ \rightarrow x \ does \ not \ belong \ to \ A$$

For set A and B on X, we also have:

$$A \subset B \ \rightarrow A \ is \ fully \ contained \ in \ B (if \ x \in A, then \ x \in B)$$
$$A \subseteq B \rightarrow A \ is \ contained \ in \ or \ is \ equivalent \ to \ B$$

We define the null set, ∅, as the set containing no elements, and the whole set, X, as the set of all elements in the universe. The null set is analogous to an impossible event, and the whole set is analogous to a certain event. All possible sets of X constitute a special set called the power set, P(X). For a specific universe X, the power set P(X) is enumerated in the following example.

Example 4.1. We have a universe comprised of three elements, X = {*a*, *b*, *c*}, so the cardinal number is $n_x = 3$. The power set is
P(X) = {∅, {*a*}, {*b*}, {*c*}, {*a, b*}, {*a, c*}, {*b, c*}, {*a, b, c*}}
The cardinality of the power set, denoted $n\mathrm{P}(X)$, is found as
$n\mathrm{P}(X) = 2^{nX} = 2^3 = 8$
Note that if the cardinality of the universe is infinite, then the cardinality of the power set is also infinity, i.e., $nX = \infty \Rightarrow n\mathrm{P}(X) = \infty$.

Operations on Classical Sets

Let A and B be two sets on the universe X. The union between the two sets, denoted A ∪ B, represents all those elements in the universe that reside in (or belong to) the set A, the set B, or both sets A and B. (This operation is also called the *logical or*; another form of the union is the *exclusive or* operation. The *exclusive or* will be described in Chapter 5.) The intersection of the two sets, denoted A ∩ B, represents all those elements in the universe X that simultaneously reside in (or belong to) both sets A and B. The complement of a set A, denoted A, is defined as the collection of all elements in the universe that do not reside in the set A. The difference of a set A with respect to B, denoted A | B, is defined as the collection of all elements in the universe that reside in A and that do not reside in B simultaneously. These operations are shown below in set-theoretic terms.

Union A ∪ B = {x | x ∈ A or x ∈ B} (4.1)

Intersection A ∩ B = {x | x ∈ A and x ∈ B} (4.2)

Complement A = {x | x _∈ A, x ∈ X} (4.3)

Difference A | B = {x | x ∈ A and x _∈ B} (4.4)

These four operations are shown in terms of Venn diagrams in Figs. 4.4–4.5.

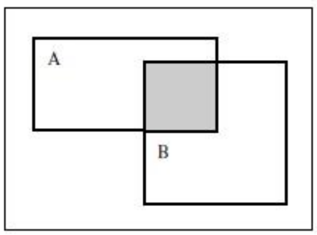


FIGURE 4.2: Union of sets A and B (logical or).

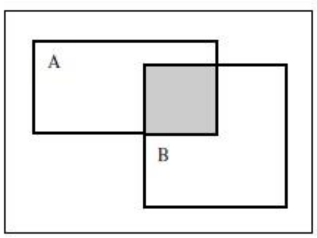


FIGURE 4.3: Intersection of sets A and B.

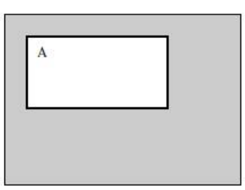


FIGURE 4.4: Complement of set A.

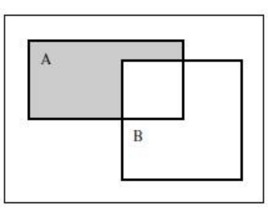


FIGURE 4.5: Difference operation A | B.

Properties of Classical (Crisp) Sets

Certain properties of sets are important because of their influence on the mathematical manipulation of sets. Themost appropriate properties for defining classical sets and showing their similarity to fuzzy sets are as follows:

Commutativity	$A \cup B = B \cup A$	
	$A \cap B = B \cap A$	(4.5)
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$	
	$A \cap (B \cap C) = (A \cap B) \cap C$	(4.6)
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(4.7)
Idempotency	$A \cup A = A$	
	$A \cap A = A$	(4.8)
Identity	$A \cup \emptyset = A$	
	$A \cap X = A$	
	$A \cap \emptyset = \emptyset$	(4.9)
	$A \cup X = X$	
Transitivity	If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$	(4.10)
Involution	$A = A$	(4.11)

The double-cross-hatched area in Fig. 2.6 is a Venn diagram example of the associativity property for intersection, and the double-cross-hatched areas in Figs. 4.7 and 4.8

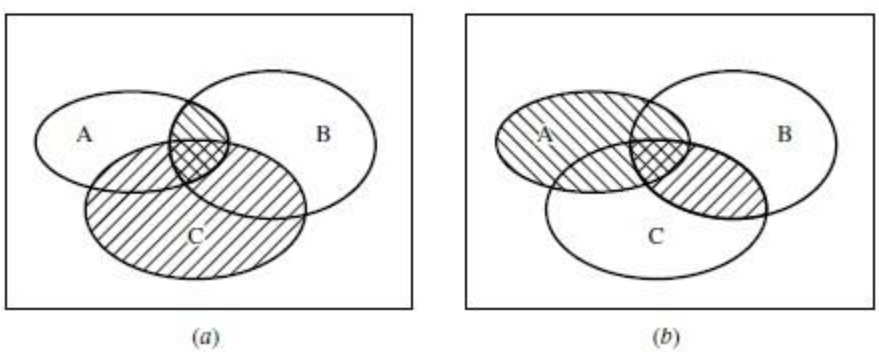


FIGURE 4.6: Venn diagrams for (*a*) $(A \cap B) \cap C$ and (*b*) $A \cap (B \cap C)$

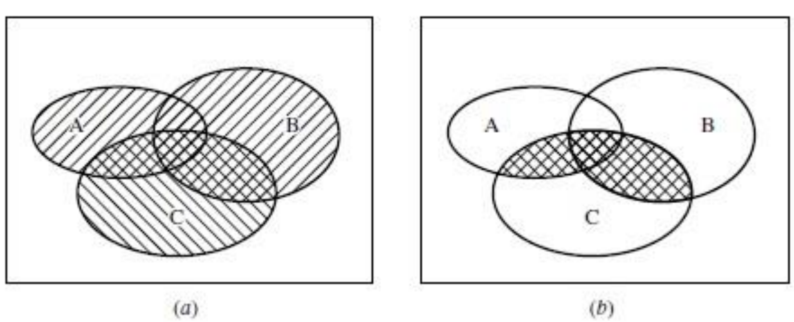


FIGURE 4.7: Venn diagrams for (*a*) $(A \cup B) \cap C$ and (*b*) $(A \cap C) \cup (B \cap C)$.

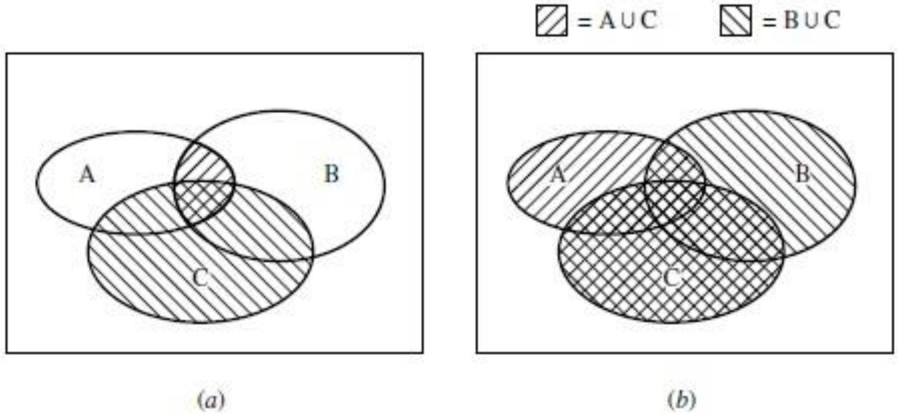


FIGURE 4.8: Venn diagrams for (*a*) *(*A ∩ B*)* ∪ C and (*b*) *(*A ∪ C*)* ∩ *(*B ∪ C*)*.

are Venn diagram examples of the distributivity property for various combinations of the intersection and union properties.

Two special properties of set operations are known as the *excluded middle axioms* and *De Morgan's principles*. These properties are enumerated here for two sets A and B.

The *excluded middle axioms* are very important because these are the only set operations described here that are *not* valid for both classical sets and fuzzy sets. There are two excluded middle axioms (given in Eqs. (4.12)). The first, called the *axiom of the excluded middle,* deals with the union of a set A and its complement; the second, called the *axiom of contradiction*, represents the intersection of a set A and its complement.

Axiom of the excluded middle $A \cup A = X$ *(4.12a)*

Axiom of the contradiction $A \cap A = \emptyset$ *(4.12b)*

De Morgan's principles are important because of their usefulness in proving tautologies and contradictions in logic, as well as in a host of other set operations and proofs. De Morgan's principles are displayed in the shaded areas of the Venn diagrams in Figs. 4.9 and 4.10 and described mathematically in Eq. (4.13).

$A \cap B = A \cup B$ (4.13*a*)

$A \cup B = A \cap B$ (4.13*b*)

In general, De Morgan's principles can be stated for *n* sets, as provided here for events, E*i* :

$E1 \cup E2 \cup \cdots \cup En = E1 \cap E2 \cap \cdots \cap En$ (4.14*a*)

$E1 \cap E2 \cap \cdots \cap En = E1 \cup E2 \cup \cdots \cup En$ (4.14*b*)

From the general equations, Eqs. (4.14), for De Morgan's principles we get a duality relation: the complement of a union or an intersection is equal to the intersection or union, respectively, of the respective complements. This result is very powerful in dealing with

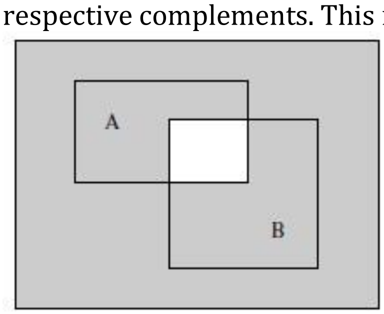


FIGURE 4.9: De Morgan's principle *(*A ∩ B*)*.

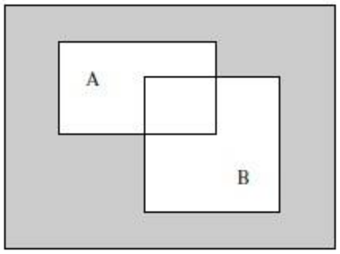


FIGURE 4.10: De Morgan's principle *(*A ∪ B*)*.

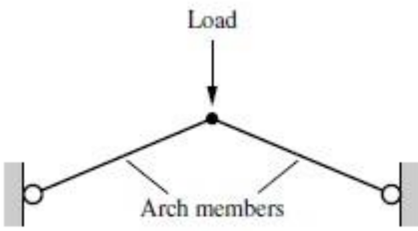


FIGURE 4.11: A two-member arch

set structures since we often have information about the complement of a set (or event), or the complement of combinations of sets (or events), rather than information about the sets themselves.

Example 4.2. A shallow arch consists of two slender members as shown in Fig. 4.11.
If either member fails, then the arch will collapse. If E1 = survival of member 1 and E2 = survival of member 2, then survival of the arch = E1 ∩ E2, and, conversely, collapse of the arch = E1 ∩ E2. Logically, collapse of the arch will occur if either of the members fails,
i.e., when E1 ∪ E2. Therefore,
E1 ∩ E2 = E1 ∪ E2
which is an illustration of De Morgan's principle.

Now, define two sets, A and B, on the universe X. The union of these two sets in terms of function-theoretic terms is given as follows (the symbol ∨ is the maximum operator and ∧ is the minimum operator):

Union $A \cup B \longrightarrow \chi_{A \cup B}(x) = \chi_A(x) \vee \chi_B(x) = \max(\chi_A(x), \chi_B(x))$ (4.16)

The intersection of these two sets in function-theoretic terms is given by
Intersection $A \cap B \longrightarrow \chi_{A \cap B}(x) = \chi_A(x) \wedge \chi_B(x) = \min(\chi_A(x), \chi_B(x))$ (4.17)

The complement of a single set on universe X, say A, is given by
Complement $A \longrightarrow \chi_A(x) = 1 - \chi_A(x)$ (4.18)

For two sets on the same universe, say A and B, if one set (A) is contained in another set
(B), then

Containment $A \subseteq B \longrightarrow \chi_A(x) \le \chi_B(x)$ (4.19)

FUZZY SETS
In classical, or crisp, sets the transition for an element in the universe between membership and nonmembership in a given set is abrupt and well-defined (said to be "crisp"). For an element in a universe that contains fuzzy sets, this transition can be gradual. This transition among various

degrees of membership can be thought of as conforming to the fact that the boundaries of the fuzzy sets are vague and ambiguous. Hence, membership of an element from the universe in this set is measured by a function that attempts to describe vagueness and ambiguity.
A fuzzy set, then, is a set containing elements that have varying degrees of membership in the set. This idea is in contrast with classical, or crisp, sets because members of a crisp set would not be members unless their membership was full, or complete, in that set (i.e., their membership is assigned a value of 1). Elements in a fuzzy set, because their membership need not be complete, can also be members of other fuzzy sets on the same universe.
Elements of a fuzzy set are mapped to a universe of *membership values* using a function-theoretic form. Fuzzy sets are denoted in this text by a set symbol with a tilde understrike; so, for example, $\underline{A}$ would be the *fuzzy set A*.
This function maps elements of a fuzzy set $\underline{A}$ to a real numbered value on the interval 0 to 1.
A notation convention for fuzzy sets when the universe of discourse, X, is discrete and finite, is as follows for a fuzzy set $\underline{A}$

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \cdots \right\} = \left\{ \sum_i \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i} \right\} \qquad 2.20$$

When the universe, X, is continuous and infinite, the fuzzy set A∼ is denoted by

$$\underset{\sim}{A} = \left\{ \int \frac{\mu_{\underset{\sim}{A}}(x)}{x} \right\} \qquad 2.21$$

In both notations, the horizontal bar is not a quotient but rather a delimiter or support. The numerator
in each term is the membership value in set A∼ associated with the element of the universe

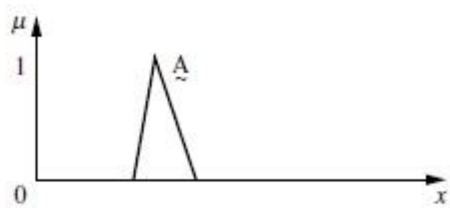


FIGURE 4.14: Membership function for fuzzy set $\underline{A}$

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indicated in the denominator. In the first notation, the summation symbol is not for algebraic summation but rather denotes the collection or aggregation of each element; hence the "+" signs in the first notation are not the algebraic "add" but are an aggregation or collection operator. In the second notation the integral sign is not an algebraic integral but a continuous function-theoretic aggregation operator for continuous variables.

Fuzzy Set Operations

Define three fuzzy sets $\underline{A}$, *B* and C on the universe X. For a given element *x* of the universe, the following function-theoretic operations for the set-theoretic operations of union, intersection, and complement are defined forA∼, B∼, and C∼ on X:

Union $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ 4.22

Intersection $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ 4.23

Complement $\mu_{\bar{A}}(x) = 1 - \mu_{A}(x)$ 4.24
(x) = 1 − *μ*A∼
*(x) (*2.24*)*
The operations given in Eqs. (4.22)–(4.24) are known as the *standard fuzzy operations*. There are many other fuzzy operations, and a discussion of these is given later in this chapter.
Any fuzzy setA∼ defined on a universeX is a subset of that universe. Also by definition, just as with classical sets, the membership value of any element *x* in the null set ∅ is 0,

To illustrate these ideas numerically, let's say we have two discrete fuzzy sets, namely

$$\underset{\sim}{A} = \left\{\frac{1}{2} + \frac{0.5}{3} + \frac{0.3}{4} + \frac{0.2}{5}\right\} \quad \text{and} \quad \underset{\sim}{B} = \left\{\frac{0.5}{2} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$$

We can now calculate several of the operations just discussed (membership for element 1 in both $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is implicitly 0):

Complement

$$\overline{\underset{\sim}{A}} = \left\{\frac{1}{1} + \frac{0}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5}\right\}$$

$$\overline{\underset{\sim}{B}} = \left\{\frac{1}{1} + \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.8}{4} + \frac{0.6}{5}\right\}$$

Union

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \left\{\frac{1}{2} + \frac{0.7}{3} + \frac{0.3}{4} + \frac{0.4}{5}\right\}$$

Intersection

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \left\{\frac{0.5}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.2}{5}\right\}$$

Difference

$$\underset{\sim}{A} \mid \underset{\sim}{B} = \underset{\sim}{A} \cap \overline{\underset{\sim}{B}} = \left\{\frac{0.5}{2} + \frac{0.3}{3} + \frac{0.3}{4} + \frac{0.2}{5}\right\}$$

$$\underset{\sim}{B} \mid \underset{\sim}{A} = \underset{\sim}{B} \cap \overline{\underset{\sim}{A}} = \left\{\frac{0}{2} + \frac{0.5}{3} + \frac{0.2}{4} + \frac{0.4}{5}\right\}$$

De Morgan's principles

$$\overline{\underset{\sim}{A} \cup \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cap \overline{\underset{\sim}{B}} = \left\{\frac{1}{1} + \frac{0}{2} + \frac{0.3}{3} + \frac{0.7}{4} + \frac{0.6}{5}\right\}$$

$$\overline{\underset{\sim}{A} \cap \underset{\sim}{B}} = \overline{\underset{\sim}{A}} \cup \overline{\underset{\sim}{B}} = \left\{\frac{1}{1} + \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} + \frac{0.8}{5}\right\}$$

TUTORIAL QUESTIONS

FUZZY SETS AND FUZZY LOGIC

1. You are asked to select an implementation technology for a numerical processor. Computation throughput is directly related to clock speed. Assume that all implementations will be in the same family (e.g., CMOS). You are considering whether the design should be implemented using medium-scale integration (MSI) with discrete parts, field-programmable array parts (FPGA), or multichip modules (MCM). Define the universe of potential clock frequencies as X = {1, 10, 20, 40, 80, 100} MHz; and define MSI, FPGA, and MCM as fuzzy sets of clock frequencies that should be implemented in each of these technologies, where the following table defines their membership values:

Clock Frequency MHz	MSI	FPGA	MCM
1	1	0.3	0
10	0.7	1	0
20	0.4	1	0.5
40	0	0.5	0.7
80	0	0.2	1
100	0	0	1

Representing the three sets as MSI = M ∼, FPGA = F∼, and MCM= C∼, find the following:

(a) M ∪ F
(b) M∩ F
(c) $\overline{M}$
(d) $\overline{F}$
(e) C ∩ $\overline{F}$
(f) $\overline{M \cap C}$

2. An engineer is asked to develop a glass break detector/discriminator for use with residential alarm systems. The detector should be able to distinguish between the breaking of a pane of a glass (a window) and a drinking glass. From analysis it has been determined that the sound of a shattering window pane contains most of its energy at frequencies centered about 4 kHz whereas the sound of a shattering drinking glass contains most of its energy at frequencies centered about 8 kHz. The spectra of the two shattering sounds overlap. The membership functions for the window pane and the glass are given as $\mu_A(x)$∼ and $\mu_B(x)$∼, respectively. Illustrate the basic operations of union, intersection, complement, and difference for the following membership functions:
x = 0, 1…10 and σ = 2, when μ_A = 4 and μ_B = 8

$$\mu_A(x) = \exp\left[\frac{(x-\mu_A)^2}{2\sigma^2}\right]$$

$$\mu_B(x) = \exp\left[\frac{(x-\mu_B)^2}{2\sigma^2}\right]$$